THE DEFECT OF BENNEQUIN-ELIASHBERG INEQUALITY AND BENNEQUIN SURFACES

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Abstract. For a null-homologous transverse link $\mathcal{T}$ in a general contact manifold with an open book, we explore strongly quasipositive braids and Bennequin surfaces. We define the defect $\delta(\mathcal{T})$ of the Bennequin-Eliashberg bound.

We study relations between $\delta(\mathcal{T})$ and minimal genus Bennequin surfaces of $\mathcal{T}$. In particular, in the disk open book case, under some large fractional Dehn twist coefficient assumption, we show that $\delta(\mathcal{T}) = N$ if and only if $\mathcal{T}$ is the boundary of a Bennequin surface with exactly N negatively twisted bands. That is, the Bennequin bound is sharp if and only if it is the closure of a strongly quasipositive braid.

1. Introduction

Let B_n be the n-strand braid group with the standard generator $\sigma_1, \dots, \sigma_{n-1}$. For $1 \leq i < j \leq n$, let $\sigma_{i,j}$ be the n-braid given by

$$\sigma_{i,j} = (\sigma_{j-1}\sigma_{j-2} \cdots \sigma_{i+1})\sigma_i(\sigma_{j-1}\sigma_{j-2} \cdots \sigma_{i+1})^{-1}$$

In particular, $\sigma_{i,i+1} = \sigma_i$. The braid $\sigma_{i,j}$ (resp. $\sigma_{i,j}^{-1}$) can be understood as the boundary of a positively (resp. negatively) twisted band attached to the i-th and the j-th strands (see Figure 1). The elements in the set $\{\sigma_{i,j}\}_{1 \leq i < j \leq n}$ are called the *band generators*.

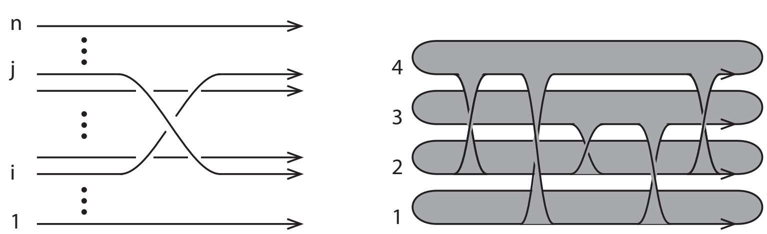

Figure 1. (Left) The band generator $\sigma_{i,j}$. (Right) The Bennequin surface associated to $\sigma_{2,4}^{-1}\sigma_{1,4}\sigma_{2,3}^{-1}\sigma_{1,3}\sigma_{2,4}^{-1}$.

Band generators appear in many papers in the literature. The work of Bennequin in [2] identifies braid words in band generators and transverse knots and links (in the following we just say link for simplicity) in the standard tight contact 3-sphere (S^3, ξ_{std}).

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Rudolph uses band generators in a series of works including [32] where he develops and popularizes the concepts of quasipositive and strongly quasipositive knots and links. See also Rudolph's survey article [33].

Using band generators, Xu in [34] gives a new presentation of B_3 and a new solution to the conjugacy problem in B_3. Birman, Ko and Lee in [4] generalize the results of Xu to B_n. From the modern viewpoint, their work can be understood that the band generators give rise to a Garside structure, which is a certain combinatorial structure allowing us to solve various decision problems like the word and conjugacy problem (see [8]). Today the Garside structure defined by band generators is called the *dual Garside structure* on B_n.

In this paper, otherwise stated, every braid word w is in the band generators $\sigma_{i,j}$, rather than in the standard Artin generators $\sigma_1, \ldots, \sigma_{n-1}$.

Let $\mathcal{T}$ be a transverse link in (S^3, ξ_{std}). We say that a word w in the band generators $\sigma_{i,j}$ is a *braid word representative* of $\mathcal{T}$ if the closure of the n-braid w is $\mathcal{T}$. For a braid word representative w of $\mathcal{T}$, starting with n disjoint disks and attaching a twisted band for each $\sigma_{i,j}^{\pm 1}$ in the word w we get a Seifert surface $F = F_w$ of $\mathcal{T}$, which we call the *Bennequin surface* associated to w (see Figure 1).

A Bennequin surface is defined by Birman and Menasco in [6, p.71] for a *topological* link type which generalizes Bennequin's Markov surface [2] (every Bennequin surface is a Markov surface, but there are Markov surfaces which are not Bennequin surfaces [6, p.73]), where they require one more additional condition that F has *maximal* Euler characteristic among all Seifert surfaces. However, in this paper, F_w may not necessarily realize the maximal Euler characteristic.

A braid $K \in B_n$ is called *strongly quasipositive* [33] if K admits a word representative w such that its associated Bennequin surface F_w has no negatively twisted bands. That is, w is a product of positive band generators. Using the dual Garside structure on B_n with the band generators, one can check whether a given braid K is conjugate to a strongly quasipositive braid or not [4].

Bennequin in [2] shows that for a braid word representative w of $\mathcal{T}$, the self-linking number $sl(\mathcal{T})$ is given by the formula

$$sl(\mathcal{T}) = -n(w) + \exp(w) \tag{1.1}$$

where $n(w)$ and $\exp(w)$ denote the number of braid strands and the exponent sum of w. He also proves a fundamental inequality called the *Bennequin inequality* [2]

$$sl(\mathcal{T}) \leq -\chi(\mathcal{T}) := 2g(\mathcal{T}) - 2 + |\mathcal{T}|, \tag{1.2}$$

where $g(\mathcal{T})$ denotes the 3-genus (of the topological link type) of $\mathcal{T}$ and $|\mathcal{T}|$ denotes the number of link components of $\mathcal{T}$. The topological invariant $\chi(\mathcal{T})$ is called the *Euler characteristic of* $\mathcal{T}$. We note that in general

$$\chi(\mathcal{T}) \geq \chi(F_w). \tag{1.3}$$

Definition 1.1. To measure how far the Bennequin inequality (1.2) is from the equality, we define the *defect of the Bennequin bound* for a transverse link $\mathcal{T}$ by

$$\delta(\mathcal{T}) := \frac{1}{2}\big(-\chi(\mathcal{T}) - sl(\mathcal{T})\big).$$

Note that $\delta(\mathcal{T})$ is a non-negative integer.

For a braid word representative w of $\mathcal{T}$, Definition 1.1, (1.1) and (1.3) imply that

$$\begin{aligned} 0 \leq \delta(\mathcal{T}) &= \frac{1}{2}(-\chi(\mathcal{T}) - sl(\mathcal{T})) \leq \frac{1}{2}(-\chi(F_w) - sl(\mathcal{T})) \\ &= \text{the number of negatively twisted bands in } F_w. \end{aligned}$$

Therefore, we observe the following:

Observation 1.2. The genus of the Bennequin surface F_w is equal to $g(\mathcal{T})$ if and only if the number of negatively twisted bands of F_w is equal to $\delta(\mathcal{T})$. In particular, for a strongly quasipositive braid word w, its Bennequin surface F_w gives a minimum genus Seifert surface of $\mathcal{T}$ and the Bennequin bound is sharp, i.e. $\delta(\mathcal{T}) = 0$.

Related to Observation 1.2 we conjecture the following:

Let $b(\mathcal{T})$ be the *braid index* of the transverse link $\mathcal{T}$ defined by

$$b(\mathcal{T}) := \min\{n \in \mathbb{Z}_{>0} \mid \mathcal{T} \text{ has an } n\text{-braid word representative}\}.$$

Conjecture 1. *Every transverse link $\mathcal{T}$ in (S^3, ξ_{std}) is represented by a braid word w whose Bennequin surface F_w contains $\delta(\mathcal{T})$ negative bands. Equivalently, due to Observation 1.2, every $\mathcal{T}$ bounds a Bennequin surface of genus $g(\mathcal{T})$.*

In Conjecture 1, we do not require that the braid word w realizes the braid index $b(\mathcal{T})$. In fact, in [17] Hirasawa and Stoimenow gives an example $\mathcal{T}$ of $b(\mathcal{T}) = 4$ represented by

$$\mathcal{T} = \sigma_{1,2}(\sigma_{2,4})^2(\sigma_{1,2})^{-1}\sigma_{1,3}\sigma_{1,2}(\sigma_{2,4})^{-1}(\sigma_{1,2})^{-2}(\sigma_{1,3})^{-2}$$

(note the sign convention is altered here) and none of whose Bennequin surfaces consisting of four disks and twisted bands have the genus $g(\mathcal{T}) = 3$.

However, studying the open book foliation of the genus 3 surface depicted in [17, Fig 2 (b)] we can verify that one positive stabilization (cf Figure 5) of this 4-braid produces a 5-braid representative of $\mathcal{T}$ that bounds a Bennequin surface of genus $g(\mathcal{T}) = 3$ as sketched in [17, Fig 2 (d)]. Concerning the the braid index, we give the following stronger version of Conjecture 1.

(Stronger Form of Conjecture 1). *Every transverse link $\mathcal{T}$ in (S^3, ξ_{std}) is represented by a braid word w of the braid index at most $b(\mathcal{T}) + \delta(\mathcal{T})$ such that its Bennequin surface F_w contains $\delta(\mathcal{T})$ negative bands.*

Under a condition of large FDTC, Conjecture 1 holds as stated in Theorem 1.12.

A special case of Conjecture 1 where $\delta(\mathcal{T}) = 0$ is of our interest.

Conjecture 2. *For a transverse link $\mathcal{T}$ in (S^3, ξ_{std}), the Bennequin bound is sharp if and only if $\mathcal{T}$ is represented by a strongly quasipositive braid.*

(Stronger Form of Conjecture 2). *For a transverse link $\mathcal{T}$ in (S^3, ξ_{std}), the Bennequin bound is sharp if and only if $\mathcal{T}$ is represented by a strongly quasipositive braid of the braid index $b(\mathcal{T})$.*

The statement of Conjecture 2 has been existing for more than a decade as a question or as a conjecture among a number of mathematicians, including Etnyre, Hedden [16, Conjecture 40], Rudolph and Van Horn-Morris.

Under a condition on large FDTC, both Conjecture 2 and its stronger form hold as stated in Corollary 1.13.

Using Hedden's result of *topological* fibered knots [15, Theorem 1.2], we can immediately show that Conjecture 2 holds for fibered transverse knots in (S^3, ξ_{std}). More generally, Etnyre and Van Horn-Morris give a characterization of fibered transverse links in general contact 3-manifolds on which the Bennequin-Eliashberg bound (cf. Theorem 1.5) is sharp [11, Theorem 1.1].

The aim of this paper is to study these conjectures in the setting of general contact 3-manifolds.

First we recall a fundamental fact repeatedly used in this paper: In a general closed oriented contact manifold supported by an open book, every closed braid can be seen as a transverse link. Conversely, every transverse link can be represented by a closed braid, which is uniquely determined up to positive stabilizations, positive destabilizations and braid isotopy (see [2, 29] for the case of disk open book (D^2, id) and [28, 30, 31] for general case).

Next, we set up some terminologies.

Definition 1.3. Let $\mathcal{T}$ be a null-homologous transverse link in a contact 3-manifold (M, ξ). We say that $\alpha \in H_2(M, \mathcal{T}; \mathbb{Z})$ is a *Seifert surface class* if $\alpha = [F]$ for some Seifert surface F of $\mathcal{T}$. This is equivalent to $\alpha \in \partial^{-1}([\mathcal{T}])$, where $[\mathcal{T}] \in H_1(\mathcal{T}; \mathbb{Z})$ is the fundamental class of $\mathcal{T} \cong S^1 \cup \cdots \cup S^1$ and $\partial : H_2(M, \mathcal{T}; \mathbb{Z}) \to H_1(\mathcal{T}; \mathbb{Z})$ is the boundary homomorphism of the long exact sequence of the pair $(M, \mathcal{T})$. Let $sl(\mathcal{T}, \alpha)$ denote the self-linking number of $\mathcal{T}$ with respect to α. We say that a Seifert surface F of $\mathcal{T}$ is an *α-Seifert surface* if $[F] = \alpha \in H_2(M, \mathcal{T}; \mathbb{Z})$.

Definition 1.4. Let $g(F)$ be the genus of F and $\chi(F)$ be the Euler characteristic of F. We define the *genus* and the *Euler characteristic of $\mathcal{T}$ with respect to α* by

$$g(\mathcal{T}, \alpha) := \min\{g(F) \mid F \text{ is an } \alpha\text{-Seifert surface of } \mathcal{T}\},$$

$$\chi(\mathcal{T}, \alpha) := \max\{\chi(F) \mid F \text{ is an } \alpha\text{-Seifert surface of } \mathcal{T}\}.$$

We have $\chi(\mathcal{T}, \alpha) = 2 - 2g(\mathcal{T}, \alpha) - |\mathcal{T}|$, where $|\mathcal{T}|$ denotes the number of link components of $\mathcal{T}$.

We recall a theorem of Eliashberg.

Theorem 1.5 (The Bennequin-Eliashberg inequality [9])**.** *The contact manifold (M, ξ) is tight if and only if for any null-homologous transverse link $\mathcal{T}$ and its Seifert class α we have*

$$sl(\mathcal{T}, \alpha) \leq -\chi(\mathcal{T}, \alpha).$$

For an overtwisted contact manifold (M, ξ), the same inequality holds for any null-homologous, non-loose transverse link $\mathcal{T}$ and its Seifert class α.

The second statement is attributed to Światkowski and a proof can be found in Etnyre's paper [12, Proposition 1.1].

Theorem 1.5 guides us to introduce the following invariant.

Definition 1.6. We define the *defect of the Bennequin-Eliashberg bound with respect to* α by

$$\delta(\mathcal{T}, \alpha) := \frac{1}{2}\big(-\chi(\mathcal{T}, \alpha) - sl(\mathcal{T}, \alpha)\big).$$

Note that $\delta(\mathcal{T}, \alpha)$ is an integer and it can be any negative integer when ξ is overtwisted: To see this, we observe that a transverse push-off of an overtwisted disk gives an transverse unknot U bounding a disk, D, with $sl(U, [D]) = 1$ and $\delta(U, [D]) = -1$. Taking some boundary connect sum of n copies of D (with bands each of which contains one positive hyperbolic point as illustrated in Figure 5 (i)) we get a disk, D_n, with $sl(\partial D_n, [D_n]) = 2n - 1$ and $\delta(\partial D_n, [D_n]) = -n$.

In Definition 4.3, we define an *α-Bennequin surface* with respect to a general open book (S, ϕ) of a general contact 3-manifold as an α-Seifert surface admitting a disk-band decomposition adapted to the open book (S, ϕ). We say that a closed braid is *α-strongly quasipositive* if it is the boundary of an α-Bennequin surface without negatively twisted bands. We say that an α-Bennequin surface F is a *minimum genus α-Bennequin surface* if $g(F) = g(\mathcal{T}, \alpha)$.

The definition of α-strongly quasipositive has been discussed by Etnyre, Hedden, and Van Horn-Morris since around 2009. It was formally introduced by Baykur, Etnyre, Hedden, Kawamuro and Van Horn-Morris in the SQuaRE meeting at the American Institute of Mathematics in July 2015, and is printed in the official SQuaRE report [1]. Later, Ito independently came up with the same definition. Hayden also has the same definition [14, Definition 3.3].

As we will see in Lemma 4.5, if $\mathcal{T}$ bounds a minimum genus α-Bennequin surface, then $\delta(\mathcal{T}, \alpha) \geq 0$. We expect that the converse is true:

Conjecture 3. *Let (S, ϕ) be an open book decomposition of a contact 3-manifold (M, ξ). Let $\mathcal{T}$ be a null-homologous transverse link in (M, ξ) with a Seifert surface class $\alpha \in H_2(M, \mathcal{T}; \mathbb{Z})$. If $\delta(\mathcal{T}, \alpha) \geq 0$ then $\mathcal{T}$ bounds a minimum genus α-Bennequin surface with respect to (S, ϕ).*

We list evidences for Conjecture 3.

First, we show that the minimum genus Bennequin surface always exists, if we forget the contact structure and only consider the *topological* link type of $\mathcal{T}$ in M. The statement is proved by Birman and Finkelstein in [3, Theorem 4.2] for a special case where $M = S^3$ and $(S, \phi) = (D^2, id)$ the disk open book.

Theorem 1.7 (proved in Section 4.3)**.** *Let M be an oriented, closed 3-manifold with an open book decomposition (S, ϕ). For every null-homologous topological link type $\mathcal{K}$ in M and its Seifert surface class $\alpha \in H_2(M, \mathcal{K}; \mathbb{Z})$, $\mathcal{K}$ bounds a minimum genus α-Bennequin surface with respect to (S, ϕ).*

Second, in Proposition 5.3 we show that under some condition on the fractional Dehn twist coefficient, a transverse link bounds a minimum genus Seifert surface which is almost an α-Bennequin surface.

Third, recall Bennequin's [2, Proposition 3]. The following stronger statement in [6, Theorem 1] is proved by Birman and Menasco, in which a subtle gap in [2] concerning pouches is fixed: *Any minimal genus Seifert surface of a closed 3-braid is isotopic to a Bennequin surface with the same boundary.* This statement implies that Conjecture 3 holds for closed 3-braids with respect to the disk open book (D^2, id).

The following Proposition 1.8 and Theorem 1.9 motivate us to study Conjecture 3.

Due to Theorem 1.5, (M, ξ) is tight if and only if $\delta(\mathcal{T}, \alpha) \geq 0$ for all null-homologous $\mathcal{T}$ and its Seifert class α. Thus, if Conjecture 3 is true then we obtain a new formulation of tightness in terms of α-Bennequin surfaces:

Proposition 1.8 (proved in Section 4.3)**.** *Let (S, ϕ) be an open book decomposition of a contact 3-manifold (M, ξ). For every null-homologous transverse link $\mathcal{T}$ in (M, ξ) and its Seifert surface class α, we suppose that the link $\mathcal{T}$ bounds a minimum genus α-Bennequin surface with respect to (S, ϕ). Then (M, ξ) is tight.*

The converse of the above statement is true if Conjecture 3 is true. Namely, if Conjecture 3 is true and (M, ξ) is tight, then for every null-homologous transverse link $\mathcal{T}$ in (M, ξ) and every Seifert surface class $\alpha \in H_2(M, \mathcal{T}; \mathbb{Z})$, the link $\mathcal{T}$ bounds a minimum genus α-Bennequin surface with respect to (S, ϕ).

In the setting of general open books, Conjectures 1 and 2 can be extended to Conjectures $1'$ and $2'$ as below:

Conjecture $1'$. *Let $\mathcal{T}$ be a null-homologous transverse link in (M, ξ) and $\alpha \in H_2(M, \mathcal{T}; \mathbb{Z})$ be a Seifert surface class. If $\delta(\mathcal{T}, \alpha) \geq 0$ then $\mathcal{T}$ bounds an α-Bennequin surface with $\delta(\mathcal{T}, \alpha)$ negative bands with respect to (S, ϕ).*

Conjecture $2'$. *Let $\mathcal{T}$ be a null-homologous link in (M, ξ) and $\alpha \in H_2(M, \mathcal{T}; \mathbb{Z})$ be a Seifert surface class. If $\delta(\mathcal{T}, \alpha) = 0$ (we say the Bennequin-Eliashberg bound is* sharp *on $(\mathcal{T}, \alpha)$) then $\mathcal{T}$ is represented by an α-strongly quasipositive braid with respect to (S, ϕ).*

Conjecture $2'$ is raised as a question in the SQuaRE report [1]. It is also stated in [1] that a strongly quasipositive link bounds a minimal genus Bennequin surface.

We remark that for a general open book, the counterpart of the stronger form of Conjecture 2 does not hold. In Example 5.6, we see an example of transverse knot $\mathcal{T}$ with $\delta(\mathcal{T}) = 0$ which bounds a minimum genus Bennequin surface but any braid representative of the minimum braid index does not bound minimum genus Bennequin surfaces.

Theorem 1.9 (proved in Section 4.3)**.** *If Conjecture 3 is true then Conjectures $1'$ and $2'$ are true.*

Theorem 1.9 and the above mentioned 3-braid result yield the following.

Corollary 1.10. *Let $\mathcal{T}$ be a transverse link in (S^3, ξ_{std}) of the braid index $b(\mathcal{T}) = 3$ with respect to the disk open book (D^2, id). Then $\mathcal{T}$ bounds a minimal genus Bennequin surface that consists of $\delta(\mathcal{T})$ negatively twisted bands, a number of positively twisted bands, and three disks.*

In particular, the Bennequin bound is sharp on $\mathcal{T}$ if and only if the 3-braid is (braid isotopic to) a strongly quasipositive braid.

1.1. **Main results.** Our first main result Theorem 1.11 confirms Conjecture 2′ under some assumptions. Let (S, ϕ) be an open book and C be a connected component of the binding of (S, ϕ), which we will call a *binding component*. Let K be a closed braid with respect to (S, ϕ) and $c(\phi, K, C)$ be the fractional Dehn twist coefficient (FDTC) of the closed braid K with respect to the binding component C (see Section 2 for the definition).

Theorem 1.11 (proved in Section 5)**.** *Let (S, ϕ) be an open book decomposition of a contact 3-manifold (M, ξ) and $\mathcal{T}$ be a null-homologous transverse link in (M, ξ) with a Seifert surface class $\alpha \in H_2(M, \mathcal{T}; \mathbb{Z})$. Assume the following:*

- (i) *S is planar.*
- (ii) *M does not contain a non-separating 2-sphere (i.e., M does not contain an $S^1 \times S^2$ in its connected summands).*
- (iii) *$\mathcal{T}$ has a closed braid representative K with respect to (S, ϕ) which bounds an α-Seifert surface F such that:*
  - (iii-a) $g(F) = g(\mathcal{T}, \alpha)$.
  - (iii-b) *Among all the binding components of (S, ϕ) only C intersects F.*
  - (iii-c) $c(\phi, K, C) > 1$.

Then $\delta(\mathcal{T}, \alpha) = 0$ if and only if K is α-strongly quasipositive with respect to (S, ϕ). In particular, $\delta(\mathcal{T}, \alpha) = 0$ if and only if $\mathcal{T}$ is represented by an α-strongly quasipositive braid.

If we drop the assumption (i) or (iii-c), as shown in Examples 5.6 and 5.8, K may not be α-strongly quasipositive. However, we note that this does not mean failure of Conjecture 2′ since some positive stabilizations of K has a good chance to be α-strongly quasipositive.

Our second main result Theorem 1.12 (and Corollary 1.13) shows that Conjecture 3 holds for the disk open book (D^2, id) under an assumption of large FDTC.

Theorem 1.12 (proved in Section 5)**.** *Let $\mathcal{T}$ be a transverse link in (S^3, ξ_{std}). Consider the disk open book (D^2, id) of (S^3, ξ_{std}). If $\mathcal{T}$ admits a closed braid representative K such that*

$$c(id, K, \partial D^2) > \frac{\delta(\mathcal{T})}{2} + 1$$

then $\mathcal{T}$ (in fact, K itself or K with one positive stabilization) bounds a minimum genus Bennequin surface with respect to (D^2, id).

Moreover, if $\delta(\mathcal{T}) = 0$ and $c(id, K, \partial D^2) > 1$ then K is a strongly quasipositive braid.

Corollary 1.13. *Let $\mathcal{T}$ be a transverse link in (S^3, ξ_{std}). Assume that $\mathcal{T}$ is represented by a closed braid K with $c(id, K, \partial D^2) > 1$ and realizing the braid index $b(\mathcal{T})$. Then the Bennequin* *bound for $\mathcal{T}$ is sharp if and only if $\mathcal{T}$ is represented by a strongly quasipositive braid of braid index $b(\mathcal{T})$ with respect to (D^2, id). (Namely, the stronger form of Conjecture 2 holds.)*

In Example 5.9 we present examples of braids satisfying conditions in Theorem 1.12 and Corollary 1.13. In particular, our example contains many non-fibered knots which shows independency of our results from Hedden's [15].

Although it looks restrictive, the large FDTC assumption is satisfied by *almost all* braids: Indeed, given a random n-braid β and a number C, the probability that $|c(id, \widehat{\beta}, \partial D^2)| \leq C$ is zero (see [27, 20] for the precise meaning of "random").

2. The FDTC for closed braids in open books

In this section we review closed braids in open books and the FDTC for closed braids.

Let S be an oriented compact surface with non-empty boundary, and $P = \{p_1, \ldots, p_n\}$ be a (possibly empty) finite set of points in the interior of S. Let $MCG(S, P)$ (denoted by $MCG(S)$ if P is empty) be the mapping class group of the punctured surface $S \setminus P$; that is, the group of isotopy classes of orientation-preserving homeomorphisms on S, fixing ∂S point-wise and fixing P set-wise.

With respect to a connected boundary component C of S, the fractional Dehn twist coefficient (FDTC) of $\phi \in MCG(S, P)$, defined in [18], is a rational number $c(\phi, C)$ and measures to how much the mapping class ϕ twists the surface near the boundary C.

Let (M, ξ) be a closed oriented contact 3-manifold (M, ξ) compatible with an open book decomposition (S, ϕ). Let $B \subset M$ be the binding of the open book and $\pi : M \setminus B \to S^1 = [0, 1]/(0 \sim 1)$ be the associated fibration. For $t \in S^1$ we denote the fiber $\pi^{-1}(\{t\})$ by S_t and call it a *page*.

A *closed braid* K with respect to (S, ϕ) is an oriented link in $M \setminus B$ which is positively transverse to each page. Two closed braids are called *braid isotopic* if they are isotopic through closed braids. The number of intersection points of K and the page S_t is denoted by $n(K)$ and called the *braid index* of K.

Let $B_n(S)$ be the n-stranded surface braid group for S. Cutting $M \setminus B$ along the page S_0 we get a cylinder $\mathrm{int}(S) \times (0, 1)$ and the closed braid K gives rise to a surface braid $\beta_K \in B_n(S)$ with $n = n(K)$ strands.

The converse direction; namely, obtaining a closed braid from a surface braid $\beta \in B_n(S)$, requires more care. Recall the generalized Birman exact sequence [13, Theorem 9.1]

$$1 \longrightarrow B_n(S) \xrightarrow{i} MCG(S, P) \xrightarrow{f} MCG(S) \longrightarrow 1 \tag{2.1}$$

where i is the push map and f is the forgetful map. Except for some special few cases, this exact sequence does not split. Since there is no canonical map from $MCG(S)$ to $MCG(S, P)$ we have various possibility to construct a closed braid K from a given braid $\beta \in B_n(S)$.

We recall the definition in [22] of the FDTC $c(\phi, K, C)$ of K as follows.

Suppose that the mapping class ϕ is represented by a homeomorphism $f \in \mathrm{Homeo}^+(S)$. For a connected boundary component C of S, let us choose a collar neighborhood $\nu(C) \subset S$ of C. We may assume that f fixes $\nu(C)$ point-wise. We say that a closed braid K is *based on* C if $K \cap S_0 \subset \nu(C)$. By a braid isotopy, we can put K in a position so that K is based on C. Let $j : S \approx S \setminus \nu(C) \hookrightarrow S$ be the inclusion map. Then j induces a homomorphism $j_* : \mathrm{Homeo}^+(S) \to \mathrm{Homeo}^+(S, P)$ since $f = id$ on $\nu(C)$. This j_* further induces a homomorphism $j_* : MCG(S) \to MCG(S, P)$.

Definition 2.1. Let K be a closed braid with respect to (S, ϕ) and based on C. The *distinguished monodromy* of the closed braid K with respect to C is the mapping class

$$\phi_K = i(\beta_K) \circ j_*(\phi) \in MCG(S, P).$$

Here i denotes the push map in the generalized Birman exact sequence. The FDTC of a closed braid K with respect to C is defined by

$$c(\phi, K, C) := c(\phi_K, C).$$

Remark 2.2. Due to the dual Garside structure of the braid group B_n coming from the band generators $\sigma_{i,j}$, for each $\beta \in B_n$ we have a left canonical normal form

$$N(\beta) = \delta^N x_1 \cdots x_m.$$

Here, $\delta = \sigma_{1,2}\sigma_{2,3}, \ldots, \sigma_{n-1,n}$ and δ is called the *dual Garside element*. As a homeomorphism of a disk with n marked points evenly distributed along the boundary, δ rotates the disk by $\frac{2\pi}{n}$. The remaining $x_1, \ldots, x_m$ are certain strongly quasipositive braids called the *dual simple elements* [4]. The integer N in the normal form $N(\beta)$ is called the *infimum* of β and denoted by $\inf(\beta)$. The *infimum* of a closed braid K is defined by

$$\inf(K) = \max\{\inf(\beta) \mid \beta \text{ is a braid representative of } K\}$$

and K is strongly quasi positive if and only if $\inf(K) \geq 0$. Although both $\inf(K)$ and $c(id, K, \partial D^2)$ count the number of twists near the boundary ∂D^2 in certain ways, in general, there are no direct connection between them. For example, let

$$\beta = \sigma_{1,n}\sigma_{n,n-1}\sigma_{n-1,n-2} \cdots \sigma_{2,3}\sigma_{1,2} \in B_n$$

for $n \geq 3$ and K_m be the closure of β^m. Then $\inf(K_m) = 0$ whereas $c(id, K, \partial D^2) = m$.

3. Summary of results in open book foliations

In this section, we review properties of open book foliations that are needed to prove our main theorems. For details, see [21, 22, 23].

Let (S, ϕ) be an open book decomposition of a contact 3-manifold (M, ξ). Let K be a closed braid with respect to (S, ϕ) and F be a Seifert surface of K. With an isotopy fixing $K = \partial F$, [21, Theorem 2.5] shows that F can admit a singular foliation

$$\mathcal{F}_{ob}(F) := \{F \cap S_t \mid t \in [0, 1]\}$$

induced by the intersection with the pages of the open book and satisfying the following conditions.

($\mathcal{F}$ i): The binding B pierces F transversely in finitely many points. At each $p \in B \cap F$ there exists a disc neighborhood $N_p \subset F$ of p on which the foliation $\mathcal{F}_{ob}(N_p)$ is radial with the node p, see Figure 2-(i). We call p an *elliptic* point.

($\mathcal{F}$ ii): The leaves of $\mathcal{F}_{ob}(F)$ are transverse to $K = \partial F$.

($\mathcal{F}$ iii): All but finitely many pages S_t intersect F transversely. Each exceptional page is tangent to F at a single point that lies in the interiors of both F and S_t. In particular, $\mathcal{F}_{ob}(F)$ has no saddle-saddle connections.

($\mathcal{F}$ iv): All the tangent points of F and fibers are of saddle type, see Figure 2-(ii). We call them *hyperbolic* points.

Such a foliation $\mathcal{F}_{ob}(F)$ is called an *open book foliation* on F.

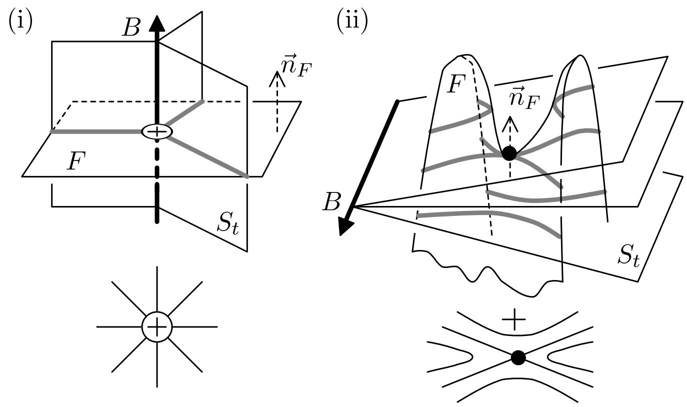

FIGURE 2. Singular points of the open book foliation and their signs. If the positive normal $\vec{n}_F$ (dashed arrow) is reversed then the sign of the singularity is multiplied by -1.

An elliptic point p is *positive* (resp. *negative*) if the binding B is positively (resp. negatively) transverse to F at p. A hyperbolic point q is *positive* (resp. *negative*) if the positive normal vector $\vec{n}_F$ of F at q agrees (resp. disagrees) with the positive normal of the page at q. We denote the sign of a singular point v by $\texttt{sgn}(v)$. See Figure 2.

A leaf of $\mathcal{F}_{ob}(F)$, a connected component of $F \cap S_t$, is called *regular* if it does not contain a hyperbolic point, and called *singular* otherwise. The regular leaves are classified into the following three types.

a-arc : An arc one of whose endpoints lies on B and the other lies on K.
b-arc : An arc whose endpoints both lie on B.
c-circle : A simple closed curve.

The leaves of $\mathcal{F}_{ob}(F)$ are equipped with orientations as follows (cf. [21, Definition 2.12]). At a non-singular point p on a leaf l in a page S_t, let $\vec{n}_S$ (resp. $\vec{n}_F$) be the positive normal vector of S_t (resp. F) at p. The positive orientation of l is determined by $\vec{n}_S \times \vec{n}_F$. With this orientation, every positive (resp. negative) elliptic point is a source (resp. sink).

According to the types of nearby regular leaves, hyperbolic points are classified into six types: Type *aa*, *ab*, *ac*, *bb*, *bc* and *cc*. Each hyperbolic point has a canonical neighborhood as depicted in Figure 3, which we call a *region*. We denote by $\mathtt{sgn}(R)$ the sign of the hyperbolic point contained in the region R. If $\mathcal{F}_{ob}(F)$ contains at least one hyperbolic point, then we can decompose F as the union of regions whose interiors are disjoint. We call such a decomposition a *region decomposition*.

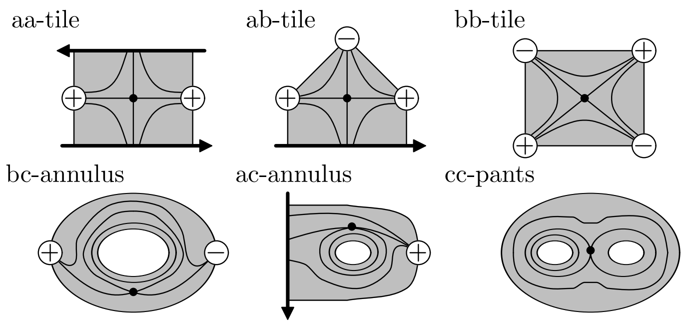

FIGURE 3. Six types of regions.

One can read the Euler characteristic and the self-linking number from the open book foliation.

Lemma 3.1. [21, Proposition 2.11, Proposition 3.2] *Let F be a Seifert surface of a transverse link $\mathcal{T}$ admitting an open book foliation $\mathcal{F}_{ob}(F)$. Let $e_\pm$ (resp. $h_\pm$) be the number of positive and negative elliptic (resp. hyperbolic) points of $\mathcal{F}_{ob}(F)$. Then the self-linking number has*

$$sl(\mathcal{T},[F]) = -(e_+ - e_-) + (h_+ - h_-).$$

For the Euler characteristics we have

$$\chi(\mathcal{T},[F]) \geq \chi(F) = (e_+ + e_-) - (h_+ + h_-).$$

Therefore, $\delta(\mathcal{T},[F]) \leq h_- - e_-$. In particular, if $g(F) = g(\mathcal{T},[F])$ then

$$\delta(\mathcal{T},[F]) = h_- - e_-.$$

We say that a b-arc b in a page S_t is *essential* if b is not boundary-parallel as an arc of the punctured page $S_t \setminus (S_t \cap K)$. We say that an open book foliation $\mathcal{F}_{ob}(F)$ is *essential* if all the b-arcs are essential (cf. [22, Definition 3.1]). The next theorem shows that an incompressible surface admits an essential open book foliation, modulo desumming essential spheres:

Theorem 3.2. [22, Theorem 3.2] *Suppose that F is an incompressible Seifert surface of a closed braid K. Then there exist a Seifert surface F' of K admitting an essential open book foliation and essential spheres $\mathcal{S}_1, \dots, \mathcal{S}_k$ such that F is isotopic to $F' \# \mathcal{S}_1 \# \cdots \# \mathcal{S}_k$ by an isotopy that fixes $K = \partial F$. Moreover, if F does not intersect a binding component C then nor does F'.*

Here is a corollary of Theorem 3.2 which we use later for the proofs of our main results.

Corollary 3.3. *Assume that M contains no non-separating 2-spheres. Let K be a closed braid representative of a null-homologous transverse link $\mathcal{T}$ and F be an incompressible Seifert surface of K. Then there is an incompressible Seifert surface F' of K with the following properties:*

- *F' admits an essential open book foliation.*
- *$[F] = [F'] \in H_2(M, \mathcal{T}; \mathbb{Z})$.*
- *$g(F') = g(F)$.*
- *If F does not intersect a binding component C then nor does F'.*

The following theorem gives a connection between essential open book foliations and the FDTC of braids.

Theorem 3.4. [22, Theorem 5.5, Theorem 5.12] *Let F be an incompressible Seifert surface of a closed braid K equipped with an essential open book foliation. Let $v_1, \ldots, v_n$ be negative elliptic points which lie on the same binding component C. Let N be the number of negative hyperbolic points that are connected to at least one of $v_1, \ldots, v_n$ by a singular leaf. Then we have*

$$c(\phi, K, C) \leq \frac{N}{n}.$$

4. Generalized Bennequin surfaces

4.1. **An algebraic formulation of Bennequin surfaces.** In this subsection, we generalize the notion of Bennequin surfaces in S^3 with the disk open book to Bennequin surfaces in general manifolds with general open books, based on an algebraic formulation. For technical reason, which we discuss at the beginning of the next subsection, we only deal with open books with connected binding.

Let (S, ϕ) be an open book with connected binding. We take an annular neighborhood $\nu = \nu(\partial S) \subset S$ of ∂S and fix an identification $\nu \approx S^1 \times [0,1]$ so that $\partial S \subset \nu$ is identified with $S^1 \times \{0\}$. Take a set of points $P = \{p_1, \ldots, p_n\}$ so that $P \subset S^1 \times \{1/2\}$. Let $\frac{1}{2}\nu := S^1 \times [0, \frac{1}{2}] \subset \nu$. See Figure 4.

We view $B_n(S)$ as a subgroup of $MCG(S, P)$ through the push map i in the generalized Birman exact sequence (2.1). We say that a braid $w \in B_n(S)$ is a *positive* (resp. *negative*) *band-twist* if $w \in MCG(S, P)$ is a positive (resp. negative) half twist about a properly embedded arc in $S \setminus \frac{1}{2}\nu$ connecting two distinct points in P (see Figure 4).

Definition 4.1. A *band-twist factorization* of a braid $\beta \in B_n(S)$ is a factorization of β into a word $w_1 \cdots w_m$, where each w_i is a band-twist. We say that β is *strongly quasipositive* if β is a product of positive band-twists.

In the case of $S = D^2$, a band-twist factorization is nothing but a factorization using the band generators $\sigma_{i,j}$.

When $g(S) > 0$, some braid in $B_n(S)$ may not admit a band-twist factorization: For example, a non-trivial 1-braid in $B_1(S) \cong \pi_1(S)$ does not admit a band-twist factorization.

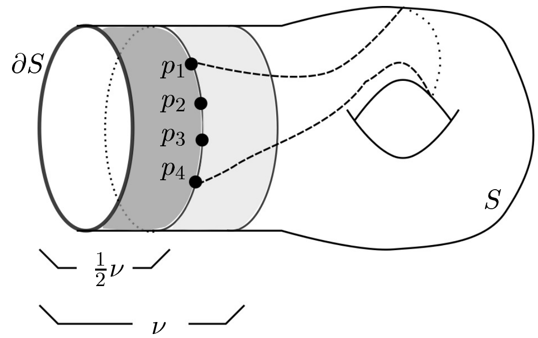

FIGURE 4. Regions $\nu, \frac{1}{2}\nu$ and an arc connecting p_1 and p_4.

For a closed braid representative K of a transverse link $\mathcal{T}$ we may put K so that $K \cap S_0 = P$. Let $\beta_K \in B_n(S)$ be the n-braid obtained by cutting M along S_0.

Definition 4.2. Suppose that β_K admits a band-twist factorization $w = w_1 \cdots w_m$. Starting with n disjoint meridional disks of the binding ∂S lying in the solid torus made of annuli pages $\frac{1}{2}\nu$, we attach a twisted band for each w_i to get a Seifert surface F_w of K. Let $\alpha := [F_w] \in H_2(M, \mathcal{T}; \mathbb{Z})$. We call this Seifert surface the *α-Bennequin surface* associated to w.

4.2. **A geometric formulation of Bennequin surfaces.** When the number of the binding components is more than one, the above algebraic formulation becomes more complicated because the generalized Birman exact sequence (2.1) is non-split as discussed in Section 2. In Definition 4.2 the open book foliation of a Bennequin surface F_w consists of only aa-tiles. We may use this as a geometric definition of an α-Bennequin surface in the general case.

Let (S, ϕ) be an open book decomposition of a contact 3-manifold (M, ξ). Let $\mathcal{T}$ be a transverse link in (M, ξ) and $\alpha \in H_2(M, \mathcal{T}; \mathbb{Z})$ be a Seifert surface class. Let K be a closed braid representative of $\mathcal{T}$ with respect to (S, ϕ).

Definition 4.3.
(1): An α-Seifert surface F of K is called an *α-Bennequin surface* of K with respect to (S, ϕ) if F admits an open book foliation whose region decomposition consists of only aa-tiles.

(2): We say that the closed braid K is *α-strongly quasipositive* with respect to (S, ϕ) if it is the boundary of an α-Bennequin surface without negative hyperbolic points. (In this case, we also say that $\mathcal{T}$ is *α-strongly quasipositive.*)

Remark 4.4. As noted in Section 1 the definition of strongly quasipositive (Definition 4.3 (2)) has been first introduced in [1, Definition 3]. Hayden independently has the same definition [14, Definition 3.3].

It is straightforward from Definition 4.3 (2) that the Bennequin-Eliashberg inequality is sharp on every α-strongly quasipositive transverse link, which is also stated in [14, Corollary 6.3].

Since an α-Bennequin surface admits an aa-tile decomposition, it is the union of disks each of which is a regular neighborhood of a positive elliptic point and twisted bands each of which is a rectangular neighborhood of a singular leaf of the open book foliation. The sign of each twisted band is equal to the sign of the corresponding hyperbolic point.

For an open book with connected binding, the above observation shows that the algebraic and the geometric definitions (Definitions 4.2 and 4.3) of an α-Bennequin surface (and strongly quasipositive) are equivalent in the following sense. The (algebraically constructed) α-Bennequin surface F_w associated with a band-twist factorization w clearly admits an aa-tile decomposition. Also every (geometrically constructed) α-Bennequin surface can be seen as F_w for some w.

4.3. **Minimum genus Bennequin surfaces.** In this section, we prove Theorems 1.9 and 1.7. We begin with a simple observation that if a transverse link is the boundary of a Bennequin surface then it satisfies the Bennequin-Eliashberg inequality.

Lemma 4.5. *Let $\mathcal{T}$ be a null-homologous transverse link and $\alpha \in H_2(M, \mathcal{T}; \mathbb{Z})$ be a Seifert surface class. If $\mathcal{T}$ is the boundary of a minimal genus α-Bennequin surface then $\delta(\mathcal{T}, \alpha) \geq 0$.*

Proof. Since any α-Bennequin surface has $e_- = 0$, Lemma 3.1 gives $\delta(\mathcal{T}, \alpha) = h_- \geq 0$. □

Proposition 1.8 and Theorem 1.9 easily follow from Lemma 3.1.

Proof of Proposition 1.8. If (M, ξ) is overtwisted, then by the Bennequin-Eliashberg inequality theorem (Theorem 1.5), there is a transverse link $\mathcal{T}$ and its Seifert surface class α such that $\delta(\mathcal{T}, \alpha) < 0$ (e.g. take a transverse push-of of the boundary of an overtwisted disk). By Lemma 4.5 such a transverse link $\mathcal{T}$ cannot bound a minimum genus α-Bennequin surface. This proves the contrapositive of the first statement of the proposition.

To see the second statement of the proposition, we assume that (M, ξ) is tight. Theorem 1.5 and the truth of Conjecture 3 imply that for any null-homologous transverse link $\mathcal{T}$ and its Seifert surface class α, $\mathcal{T}$ bounds a minimal genus α-Bennequin surface with respect to (S, ϕ). □

Proof of Theorem 1.9. Assume that $\delta(\mathcal{T}, \alpha) \geq 0$ for some $\mathcal{T}$ and α. The truth of Conjecture 3 implies that there exists an α-Bennequin surface F with $g(F) = g(\mathcal{T}, \alpha)$. Let p (resp. n) be the number of positively (resp. negatively) twisted bands in F. By a property of the geometric definition of an α-Bennequin surface, p (resp. n) is equal to the number of positive (resp. negative) hyperbolic points of the open book foliation $\mathcal{F}_{ob}(F)$. Any α-Bennequin surface has $e_- = 0$. By Lemma 3.1 $\delta(\mathcal{K}, \alpha) = h_- = n$. Thus Conjectures $1'$ and $2'$ hold. □

Next we prove Theorem 1.7, which guarantees the existence of minimum genus Bennequin surfaces for every *topological* link type.

Let C be a connected component of the binding of the open book (S, ϕ). Let μ_C be a meridian of C whose orientation is induced from that of C. We say that a closed braid K' is a *positive* (resp. *negative*) *stabilization* of a closed braid K about C, if K' is the connect sum of μ_C and K with a positively (resp. negatively) twisted band. See Figure 5 (i). Here, a positively (resp. negatively) twisted band is a rectangle whose open book foliation has a unique positive (resp. negative) hyperbolic point.

Both positive and negative stabilizations preserve the topological link type of the closed braid K. A positive stabilization preserves the transverse link type of K, whereas a negative stabilization does not. Recall the fact that one can remove an ab-tile by a stabilization, see Figure 5 (ii) and [7, Figure 26].

Lemma 4.6. *Let K be a closed braid with respect to (S, ϕ) and F be a Seifert surface of K admitting an open book foliation. Assume that the region decomposition of F has an ab-tile R. Let C denote the binding component on which the negative elliptic point of R lies. If* $\texttt{sgn}(R) = +1$ *(resp.* -1*) then a* negative *(resp.* positive*) stabilization of K about C can remove the ab-tile R, without changing the rest of the open book foliation of F.*

Proof. We push $K = \partial F$ across the unstable separatrix of the hyperbolic point in R. See Figure 5 (ii). Then the resulting closed braid is a stabilization of K about C and the sign of stabilization is positive (resp. negative) if $\texttt{sgn}(R) = -1$ (resp. $\texttt{sgn}(R) =+1$). ☐

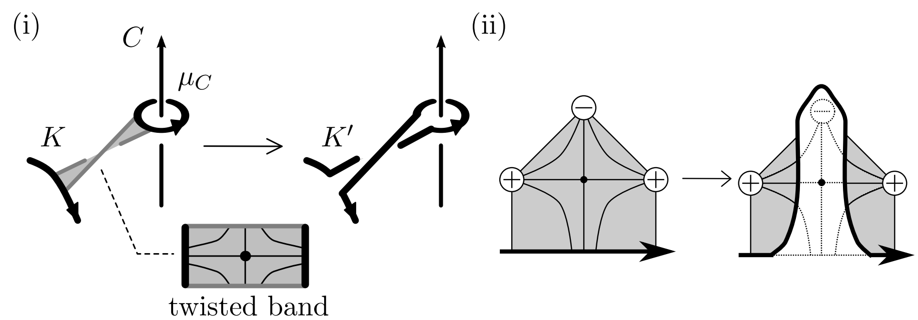

FIGURE 5. (i) Stabilization about the binding component C. (ii) Stabilization along an ab-tile.

Proof of Theorem 1.7. Take a closed braid representative K of a null-homologous topological link type $\mathcal{K}$. Let F be an α-Seifert surface of K with $g(F) = g(\mathcal{K}, \alpha)$. By an isotopy fixing K we may put F in a position so that F admits an open book foliation $\mathcal{F}_{ob}(F)$. By [21, Proposition 2.6] we may assume that $\mathcal{F}_{ob}(F)$ contains no c-circles.

By Lemma 4.6, after sufficiently many positive and negative stabilizations, we can remove all the ab-tiles without producing new c-circles. This would make an existing bb-tile becomes an ab-tile. Then we remove the new ab-tile as well by another stabilization. After removing all the ab-tiles and bb-tiles, the region decomposition consists of only aa-tiles; thus, we obtain an α-Bennequin surface. ☐

5. PROOFS OF THE MAIN THEOREMS

The goal of this section is to prove the main results (Theorems 1.11 and 1.12 and Corollary 1.13).

5.1. **Lemmas for the main results.** Let F be a Seifert surface of a closed braid K with respect to (S,ϕ). Assume that F admits an open book foliation $\mathcal{F}_{ob}(F)$. Fix a region decomposition of $\mathcal{F}_{ob}(F)$. To relate the open book foliation and the FDTC, we use the following graph $\widehat{G_{--}}$ which is a slight modification of the graph G_{--} introduced in [21, Definition 2.17].

Definition 5.1. Let R be an ab-tile, a bb-tile or a bc-annulus in the region decomposition of $\mathcal{F}_{ob}(F)$. If $\texttt{sgn}(R)=-1$ then the graph G_R on R is as illustrated in Figure 6. If $\texttt{sgn}(R)=+1$ then G_R is defined to be empty. Also, if R is an aa-tile, an ac-annulus or a cc-pants then G_R is defined to be empty. The union of graphs G_R over all the regions of the region decomposition and all the negative elliptic points gives a (possibly not connected) graph, $\widehat{G_{--}}$, contained in F. We call the graph $\widehat{G_{--}}$ the *extended graph* of G_{--}.

There are two types of vertices in $\widehat{G_{--}}$. We say that a vertex of $\widehat{G_{--}}$ is *fake* if it is not a negative elliptic point as depicted with a hollow circle in Figure 6. A negative elliptic point is called a *non-fake* vertex.

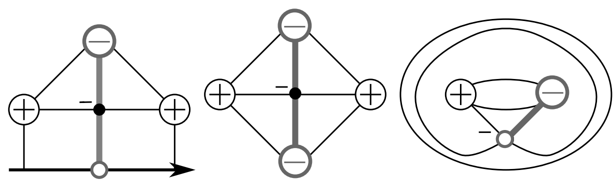

FIGURE 6. The graph G_R. A hollow circle represents a fake vertex. Negative hyperbolic points (black dots) are not vertices. An edge may or may not contain a black dot.

In Lemma 5.2 and Propositions 5.3, 5.4 and 5.5, we assume that K is a closed braid with respect to an open book (S,ϕ) representing a null-homologous transverse link $\mathcal{T}$, and F is a Seifert surface of K with

(i) $\delta(\mathcal{T},[F])\geq 0$,
(ii) $g(F)=g(K,[F])$ (which implies F is incompressible),
(iii) F admits an open book foliation $\mathcal{F}_{ob}(F)$ (which may be not essential).

Lemma 5.2. *If the open book foliation $\mathcal{F}_{ob}(F)$ contains negative elliptic points then the extended graph $\widehat{G_{--}}$ contains a non-fake vertex of valence less than or equal to $\delta(\mathcal{T},[F])+2$.*

Proof. First suppose that $e_-=1$. Let d denote the valence of the unique negative elliptic point of $\mathcal{F}_{ob}(F)$ (as a vertex of the extended graph $\widehat{G_{--}}$). Since $g(F)=g(K,[F])$ by Lemma 3.1

$$\delta(\mathcal{T},[F])=h_- - e_- \geq d-1.$$

Thus, $d\leq \delta(\mathcal{T},[F])+1$.

Next we assume that $e_-\geq 2$. For $i\geq 0$, let v_i be the number of vertices of $\widehat{G_{--}}$ whose valence is i and let w be the number of edges of $\widehat{G_{--}}$. Then we have $\sum_i iv_i=2w$ and $\chi=\sum_i v_i - w$, where $\chi=\chi(\widehat{G_{--}})$ is the Euler characteristic of the extended graph $\widehat{G_{--}}$.

Therefore,

$$\sum_{i\geq 2}(i-2)v_i = -2\chi + v_1 + 2v_0. \tag{5.1}$$

If there is a non-fake vertex of valence less than or equal to two, then we are done since $2 \leq \delta(\mathcal{T},[F])+2$ by the condition (i).

Suppose that every non-fake vertex has valence grater than two. (i.e., $v_0 = v_2 = 0$). Then, since every fake vertex has valence one, v_1 is equal to the number of fake vertices. By Definition 5.1 we have $h_- \geq w$ and

$$e_- = \sum_{i\geq 0} v_i - \#\{\text{fake vertices}\} = \sum_{i\geq 0} v_i - v_1.$$

By the condition (ii) and Lemma 3.1, we have

$$\delta(\mathcal{T},[F]) = h_- - e_- \geq w - e_- = w - (\sum_{i\geq 0} v_i) + v_1 = -\chi + v_1.$$

Therefore by (5.1) we get an inequality

$$\sum_{i\geq 2}(i-2)v_i \leq 2\delta(\mathcal{T},[F]) - v_1. \tag{5.2}$$

Let $j = \min\{i > 2 \mid v_i \neq 0\}$. Since $\widehat{G_{--}}$ contains at least two non-fake vertices, by (5.2)

$$(j-2)2 \leq \sum_{i\geq 2}(i-2)v_i \leq 2\delta(\mathcal{T},[F]) - v_1.$$

Therefore, $j \leq \delta(\mathcal{T},[F]) - \frac{v_1}{2} + 2 \leq \delta(\mathcal{T},[F]) + 2$. □

Propositions 5.3 and 5.4 below show that under an assumption of large FDTC and essentiality of the open book foliation, an α-Seifert surface is 'close' to an α-Bennequin surface in the sense that its open book foliation has no negative elliptic points (but may have c-circles).

Proposition 5.3. *Assume that the open book foliation $\mathcal{F}_{ob}(F)$ is essential. If $c(\phi,K,C) > \delta(\mathcal{T},[F])+2$ for every binding component C that intersects F, then $\mathcal{F}_{ob}(F)$ has no negative elliptic points (but possibly it has c-circles).*

Proof. Assume to the contrary that the braid foliation $\mathcal{F}_{ob}(F)$ has negative elliptic points. By Lemma 5.2, there exists a non-fake vertex v of $\widehat{G_{--}}$ whose valence is less than or equal to $\delta(\mathcal{T},[F])+2$. By Theorem 3.4, this implies that $c(\phi,K,C) \leq \delta(\mathcal{T},[F])+2$, which contradicts our assumption. □

If F intersects exactly one binding component then we can say more with a smaller lower bound on the FDTC.

Proposition 5.4. *Assume that the open book foliation $\mathcal{F}_{ob}(F)$ is essential and all the negative elliptic points lie on the same binding component, which we denote by C. If*

$$c(\phi,K,C) > \frac{1}{k}\delta(\mathcal{T},[F]) + 1$$

for some $k \geq 2$ then the number of negative elliptic points e_- is at most $k-1$.

Moreover, if $c(\phi, K, C) > 1$ and $\delta(\mathcal{T}, [F]) = 0$ then $e_- = h_- = 0$.

Proof. If $e_- = 0$ then we are done.

We may assume that $e_- \geq 1$. Let $N(\geq 0)$ be the number of negative hyperbolic points of type either ab, bb or bc. Not the $h_- \geq N$. Every negative hyperbolic point of type ab, bb or bc is connected to at least one negative elliptic point by a singular leaf. By Theorem 3.4 and Lemma 3.1 we have

$$\frac{1}{k}\delta(\mathcal{T}, [F]) + 1 < c(\phi, K, C) \leq \frac{N}{e_-} \leq \frac{h_-}{e_-} = \frac{\delta(\mathcal{T}, [F])}{e_-} + 1.$$

If $\delta(\mathcal{T}, [F]) > 0$ then we obtain $e_- \leq k-1$.

If $\delta(\mathcal{T}, [F]) = 0$ then we get $1 < 1$, a contradiction. Therefore in this case $e_- = 0$. □

The next proposition gives a criterion of strongly quasi-positive braids.

Proposition 5.5. *Assume the following.*

(i) *All the elliptic and hyperbolic points of $\mathcal{F}_{ob}(F)$ are positive.*
(ii) *The page S is planar.*
(iii) *Only one binding component intersects F.*

Then F is an $[F]$-Bennequin surface and K is a strongly quasipositive braid.

Proof. Let C be the unique binding component that intersects F. By the assumption (ii), if there exists a c-circle, c, in a page $S = S_t$ then c separates S into two components. Let X be the connected component of $S \setminus c$ that contains C.

Recall our orientation convention for leaves as defined in Section 3. We say that c is *coherent* with respect to C if the leaf orientation of c agrees with the boundary orientation of $c \subset \partial X$. Otherwise, we say that c is *incoherent*. For simplicity, we omit writing 'with respect to C' in the following.

By the assumption (i), there are no negative elliptic points. Therefore, the region decomposition of F consists of only aa-tiles, ac-annuli and cc-pants each of which has a positive hyperbolic point.

First, let us consider how an ac-singular point changes the types of local regular leaves.

(1) An a-arc forms a positive hyperbolic point h with itself then splits into an a-arc and a c-circle, c, see Figure 7 (1). By the assumption (iii), every a-arc starts at C. This shows that the c-circle c must be incoherent.

(2) An a-arc and a c-circle merge and form a positive hyperbolic point h. Then they become one a-arc, see Figure 7 (2). By the assumption (iii), this c-circle must be coherent.

Next, let us consider how a cc-hyperbolic point changes the types of local regular leaves.

(3) Suppose that a c-circle form a positive cc-hyperbolic point h with itself then splits into two c-circles. There are two possibilities.

(3-a) An incoherent c-circle splits into two incoherent c-circles, see Figure 7 (3-a).

(3-b) A coherent c-circle splits into one coherent c-circle and one incoherent c-circle, see Figure 7 (3-b).

(4) Suppose that two c-circles merge and form a positive cc-hyperbolic point then become one c-circle. There are two possibilities.

(4-a) Two coherent c-circles merge into one coherent c-circle, see Figure 7 (4-a).

(4-b) One coherent c-circle and one incoherent c-circle merge into one incoherent c-circle, see Figure 7 (4-b).

The above discussion shows that passing a type ac or cc positive hyperbolic point never decreases (resp. increases) the number of incoherent (resp. coherent) c-circles. For a regular page S_t ($t \in [0,1]$), let $N(t)$ be the number of incoherent c-circles in S_t. Since a type aa hyperbolic point does not affect c-circles we get $N(t) \leq N(t')$ for $t < t'$.

Our strategy is to show that all the regions in the region decomposition are of type aa; hence, F is an $[F]$-Bennequin surface.

Assume to the contrary that there exist c-circles. If no a-arcs interact with those c-circles (i.e., no ac-annuli exist), then the surface F contains a component consisting of only aa-tiles. In other words, F is disconnected, which is a contradiction.

Therefore, $\mathcal{F}_{ob}(F)$ contains ac-annuli.

If at least one ac-annuls of type (1) exists then we have an strict inequality $N(0) < N(1)$. However, the page S_1 is identified with the page S_0 by the monodromy ϕ of the open book. Since F is orientable, ϕ identifies an incoherent c-circle in S_1 with an incoherent c-circle in S_0, which means $N(0) = N(1)$. This is a contradiction.

If $\mathcal{F}_{ob}(F)$ contains an ac-annulus of type (2) then a parallel argument about the number of coherent c-circles holds and we get a contradiction.

Thus, c-circles do not exist. □

5.2. **Proofs of the main results.**

Proof of Theorem 1.11. (⇐) The statement is trivial.

(⇒) We assume $\delta(\mathcal{T}, \alpha) = 0$ and show that K is an α-strongly quasipositive braid.

By the assumptions (ii), (iii-a) and Corollary 3.3, after desumming essential spheres, we may assume that the new F (we abuse the same notation) admits an essential open book foliation $\mathcal{F}_{ob}(F)$ and the binding component C in the condition (iii-b) is still the only binding component that intersects the new F.

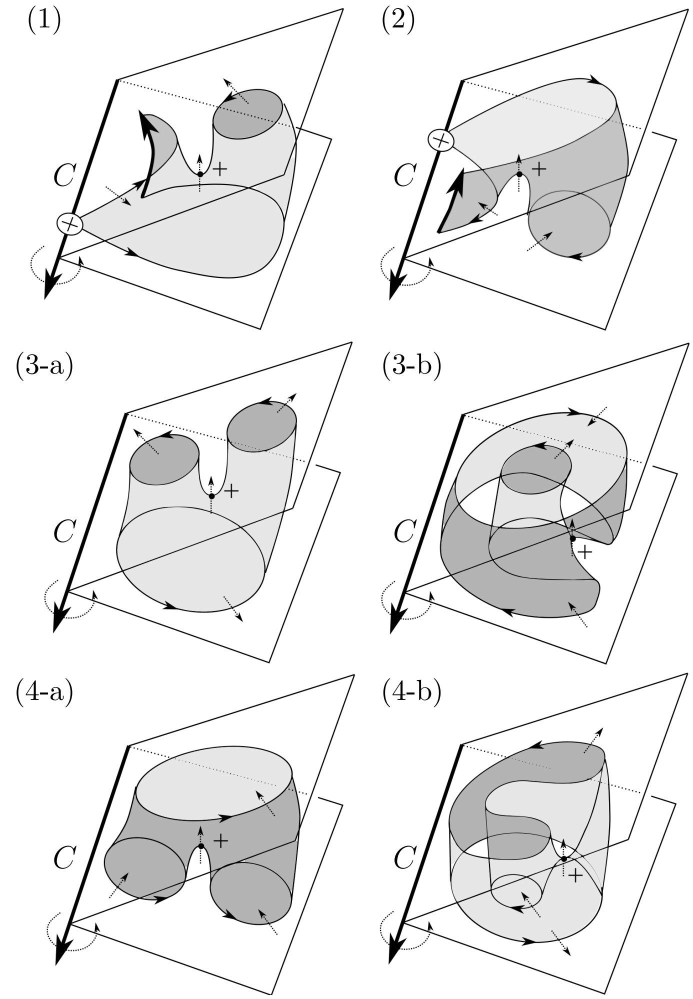

Figure 7. Positive hyperbolic points and (in)coherent c-circles. An ac or cc positive hyperbolic point does not increase (resp. decrease) the number of coherent (resp. incoherent) c-circles.

The assumptions $\delta(\mathcal{T},\alpha) = 0$ and (iii-c) give $c(\phi, K, C) > 1 = \frac{\delta(\mathcal{T},\alpha)}{2} + 1$. Hence, by Proposition 5.4, $\mathcal{F}_{ob}(F)$ has at most one negative elliptic point; that is $e_- \leq 1$.

Assume to the contrary that $e_- = 1$. Note that by (iii-b), C is the binding component on which the unique negative elliptic point lies. Then by Lemma 3.1 and the assumption (iii-a) we have $h_- = \delta(\mathcal{T},\alpha) + e_- = 0 + 1 = 1$. By Theorem 3.4 this shows that $c(\phi, K, C) \leq 1$, which contradicts the condition (iii-c).

Therefore, $e_- = h_- = 0$. By Proposition 5.5 and the assumptions (i) and (iii-b), F is an α-Bennequin surface with the strongly quasipositive boundary K. $\square$

Proof of Theorem 1.12. Let F be a minimum genus Seifert surface of K. By Corollary 3.3, we may assume that F admits an essential open book foliation.

In the case of disk open book (i.e., the braid foliation case), if F is incompressible then one can put F so that its open book foliation is essential without c-circles, see [5, Lemma 2] and [3, Lemmas 1.2 and 1.3]. Thus, F contains no bc-tiles and all the fake vertices (if they exist) of $\widehat{G_{--}}$ lie on K.

Since $c(id, K, \partial D^2) > \frac{\delta(\mathcal{T})}{2}+1$, Proposition 5.4 implies that $\mathcal{F}_{ob}(F)$ has at most one negative elliptic point.

If $e_- = 0$ then the region decomposition of $\mathcal{F}_{ob}(F)$ consists of only aa-tiles; that is, F is a Bennequin surface.

If $e_- = 1$ then $\mathcal{F}_{ob}(F)$ does not contain bb-tiles. Let v denote the unique negative elliptic point. All the ab-tiles of $\mathcal{F}_{ob}(F)$ meet at v. Suppose that the valence of v in the graph $\widehat{G_{--}}$ is N. Then by Theorem 3.4 we have

$$\frac{\delta(\mathcal{T})}{2} + 1 < c(id, K, \partial D^2) \leq \frac{N}{1} = N,$$

which shows that there are $N \geq 2$ negative ab-tiles meeting at v. Apply a positive stabilization along one of the negative ab-tiles (cf. Figure 5 (ii)) we may remove the negative elliptic point v. Note that the genus of the surface is preserved. As a consequence we get a Seifert surface whose region decomposition consists of only aa-tiles.

Moreover, if $\delta(\mathcal{T}) = 0$ and $e_1 = 1$ then $h_- = \delta(\mathcal{T}) + e_- = 0 + 1 = 1$. We have

$$1 = \frac{\delta(\mathcal{T})}{2} + 1 < c(id, K, \partial D^2) \leq \frac{N}{1} = N \leq h_- = 1,$$

which is a contradiction. Thus, when $\delta(\mathcal{T}) = 0$ the Seifert surface F is already a Bennequin surface without negatively twisted bands; hence, K is strongly quasipositive. □

5.3. **Examples.** We close the paper with examples related to the main results. Some of the examples are described via *movie presentations*. A movie presentation is a sequence of slices of Seifert surface by the pages S_t. See [21, p.1597] for the definition of a movie presentation.

Example 5.6. First we see that the planar condition (i) of Theorem 1.11 is necessary.

Suppose that S is an oriented genus 1 surface with connected boundary. Choose ϕ so that the the manifold $M_{(S,\phi)}$ is a rational homology sphere. The condition (ii) of Theorem 1.11 is automatically satisfied. Since the Seifert surface class is uniquely determined we may drop α- from our notation.

Take a base point near the boundary so that ϕ fixes it. Let ρ be an oriented loop at this base point as depicted in Figure 8 (1). Under the identification $B_1(S) = \pi_1(S)$ we may identify ρ with a 1-braid in the surface braid group $B_1(S)$.

For $N \geq 1$, let K_N be the closure of the 1-braid ρ^N with respect to the open book (S, ϕ). Then $c(\phi, K_N, \partial S) = N$. The condition (iii-c) is satisfied if $N > 1$.

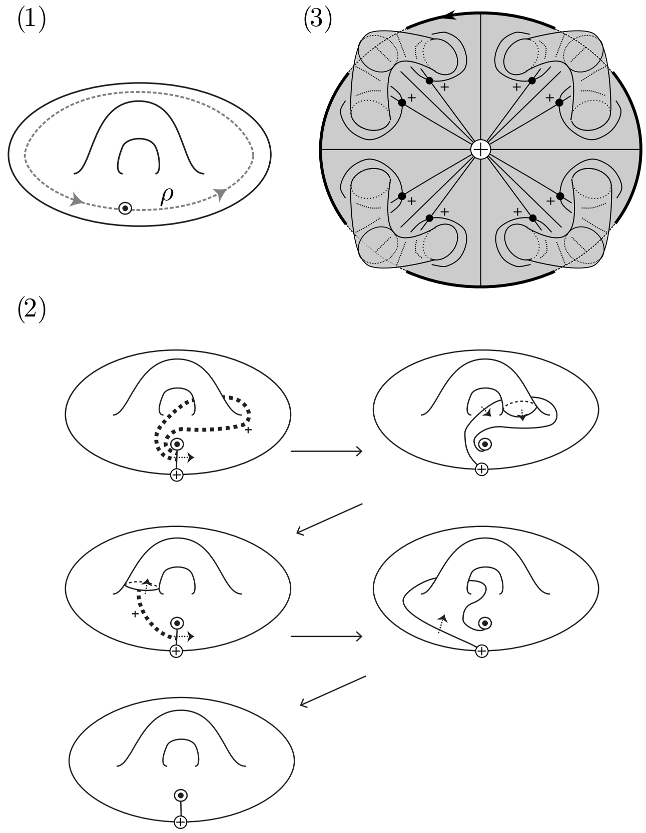

FIGURE 8. (1): The oriented loop ρ. (2) The movie presentation of the Seifert surface F for the closed braid K_N where $N = 1$. For $N \geq 2$ iterating the movie N times gives the surface F. (3) The open book foliation of F where $N = 4$.

Note that K_1 is smoothly isotopic to the binding of the open book. This shows that $g(K_1) = g(S) = 1$. Since K_N is an $(N, 1)$-cable of the binding, K_N is a connected sum of N copies of K_1, which yields $g(K_N) = N$. The Seifert surface F of K_N defined by the movie presentation in Figure 8 (2) gives a genus N surface. Therefore, the condition (iii-a) is satisfied.

The movie presentation also determines the open book foliation $\mathcal{F}_{ob}(F)$ of F as depicted in Figure 8 (3). We observe that $\mathcal{F}_{ob}(F)$ is essential (there are no b-arcs) and all the hyperbolic and elliptic points are positive. By Lemma 3.1 it follows that $\delta(K_N) = 0$.

If K_N were a strongly quasipositive braid bounding a Bennequin surface F' then due to the one-strand constraint the open book foliation $\mathcal{F}_{ob}(F')$ must be built of only a-arcs emanating from a single positive elliptic point. This means that K_N is a meridional circle of the binding; that is, an unknot. This contradicts the above conclusion $g(K_N) = N \neq 0$.

We conclude that if $N > 1$, all the conditions of Theorem 1.11 are satisfied except for the planar assumption (i) on S, and K_N is not a strongly quasipositive braid.

Example 5.7. With the above example we can further see that the stronger forms of Conjectures 1 and 2 are not true for general open books.

More concretely, we show that the transverse knot type $\mathcal{T}$ represented by the closed braid K_1 in Example 5.6 does bound a minimum genus Bennequin surface (indeed, strongly quasipositive) at the cost of raising the braid index.

To see this, we consider a different Seifert surface F' of K_1 given by a movie presentation as depicted in Figure 9. Using the Euler characteristic formula in Lemma 3.1 we see that both F and F' has genus 1, which is the genus of the transverse knot type $\mathcal{T}$. However the open book foliations of F and F' are different. For instance, the region decomposition of the open book foliations $\mathcal{F}_{ob}(F)$ consists of two ac-annuli whereas $\mathcal{F}_{ob}(F')$ consists of four ab-tiles. More precisely, $\mathcal{F}_{ob}(F')$ contains one negative elliptic point and one negative hyperbolic point and they belong to the same unique negative ab-tile.

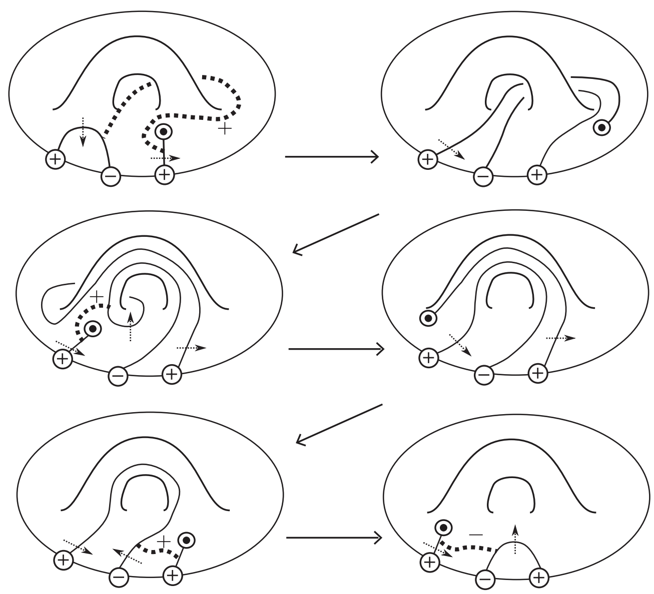

FIGURE 9. The movie presentation of the Seifert surface F' for K_1.

By a positive stabilization along the negative ab-tile we can remove both the negative elliptic and negative hyperbolic points of F'. Since any stabilization preserves the Euler characteristic of the surface, the resulting surface, F'', also has genus 1. The surface F'' consists of only positive aa-tiles and its boundary is a closed braid of braid index 2.

In summary, we obtain a minimum genus Bennequin surface F'' of $\mathcal{T}$ whose boundary is a strongly quasipositive 2-braid. Knowing that $b(\mathcal{T}) = 1$ and $\mathcal{T}$ is not an unknot, any closed 1-braid representatives of $\mathcal{T}$ are not strongly quasipositive.

For a higher $N \geq 2$, by adding extra $N-1$ pairs of positive and negative elliptic points we can construct a genus N Seifert surface for K_N where a parallel argument works, and we obtain the same conclusion.

Example 5.8. Next we see that the condition (iii-c) on the FDTC in Theorem 1.11 is also necessary.

Let S be a genus 0 surface with four boundary components C_0, C_1, C_2, C_3. Let X be a simple closed curve that separate C_1 and C_2 from C_3 and C_4. See Figure 10 (1). Let $\phi \in \mathrm{Diffeo}^+(S)$ be a diffeomorphism defined by

$$\phi = T_X T_{C_1}^{n_1} T_{C_2}^{n_2} T_{C_3}^{n_3},$$

where T_* denotes a positive Dehn twist about $* \in \{X, C_1, C_2, C_3\}$ and $n_1, n_2, n_3 \in \mathbb{Z} \setminus \{0\}$. Since $n_1, n_2, n_3 \neq 0$ the ambient manifold $M = M_{(S,\phi)}$ has $H_1(M;\mathbb{Z}) = \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \mathbb{Z}/n_3\mathbb{Z}$ (cf. [10, p.3136]) and it yields $H_2(M;\mathbb{Q}) = 0$ by the universal coefficient theorem; hence, M is a rational homology sphere and the condition (ii) of Theorem 1.11 is automatically satisfied.

The movie presentation shown in Figure 10 (3) gives a surface, which we call D. The trace of the point $\odot$ gives the boundary, K, of D. In particular, K is a 1-braid with respect to (S, ϕ). The open book foliation of D as depicted in Figure 10 (2) shows that

- D is a disk and K is an unknot ((iii-a) is satisfied).
- Among all the binding components of (S, ϕ), only C_0 intersects D ((iii-b) is satisfied).
- $c(\phi, K, C_0) = 0$.

Therefore, all the conditions of Theorem 1.11 are satisfied except for the condition (iii-c) on the FDTC. Indeed, the region decomposition of D consists of two ab-tiles and D is not even a Bennequin surface; thus K is not a strongly quasipositive braid. (We remark that after one positive stabilization, we get a strongly quasipositive braid representative of the transverse knot type $[K]$).

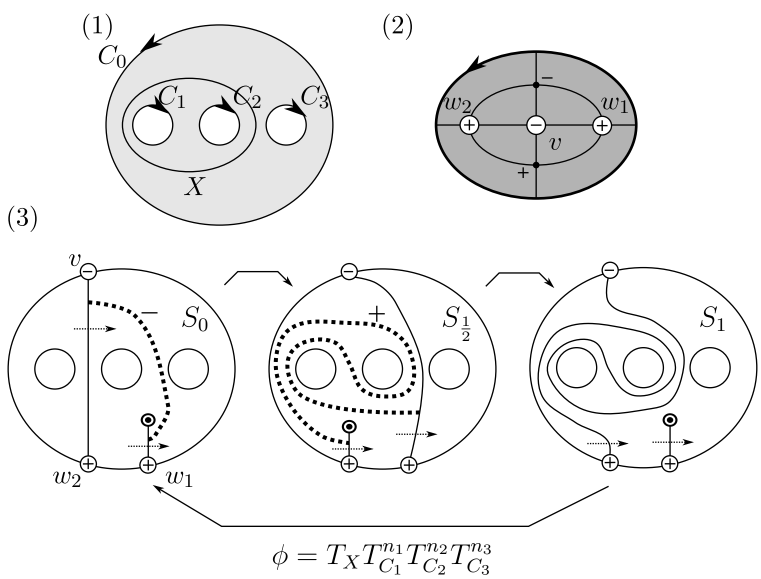

FIGURE 10. (1) The page surface S. (2) The open book foliation of D. (3) The movie presentation of D.

Example 5.9 (Example for Theorem 1.12)**.** We demonstrate that the condition $c(id, K, \partial D^2) > \frac{\delta(\mathcal{T})}{2} + 1$ in Theorem 1.12 can be satisfied by links that are neither 3-braid links or fibered knots. That is, Theorem 1.12 is independent of Corollary 1.10 and Hedden's [15, Theorem 1.2].

For a non-negative integer $\delta \geq 0$, let us consider an n-braid word of the form $w = xy$ where $x \in B_n$ is a strongly quasipositive braid word and $y \in B_n$ is a braid word containing δ negative band generators.

Let K be the closure of w and $\mathcal{T}$ be the transverse knot type represented by K. The Bennequin surface F_w associated to w has the Euler characteristic $\chi(F_w) = n - h_+ - h_- \leq \chi(\mathcal{T})$, where h_+ (resp. h_-) denotes the number of positive (resp. negative) band generators in the word w. By Lemma 3.1 we get $\delta(\mathcal{T}) \leq h_- = \delta$.

Let $c : B_n \to \mathbb{Q}$ be the FDTC map defined by $c(\beta) := c(id, \widehat{\beta}, \partial D^2)$. The map has the following properties for $\alpha, \beta \in B_n$.

(i) $|c(\alpha\beta) - c(\alpha) - c(\beta)| \leq 1$ and $c(\alpha) = c(\beta^{-1}\alpha\beta)$.
(ii) If $p \in B_n$ is a strongly quasipositive word then $c(\alpha p \beta) \geq c(\alpha\beta) \geq c(\alpha p^{-1}\beta)$ (indeed this holds for right-veering braids p [24, Corollary 3.1]).
(iii) $c(\sigma_{i,j}^{\pm 1}) = 0$.
(iv) If β is the product of m negative band generators then $c(\beta) > -\frac{m+1}{n}$.

Property (i) can be found in [26] and [22, Corollary 4.17]. Property (iii) follows from [22, Lemma 4.13]. Property (iv) follows from the proof of [19, Proposition 2.4], which is an estimate of another invariant of braids called the Dehornoy floor $[\beta]_D$, together with [22, Lemma 4.13].

By (ii) and (iv) we have $c(y) > -\frac{\delta+1}{n}$. Now let us take a strongly quasipositive braid word x such that $c(x) \geq \frac{\delta}{2} + \frac{\delta+1}{n} + 2$. (For example, $x = (\sigma_{1,n}\sigma_{1,2}\sigma_{2,3}\cdots\sigma_{n-1,n})^N$ for $N \geq \frac{1}{2}\delta + \frac{\delta+1}{n} + 2$ satisfies this condition.) By (i) we have

$$c(w) \geq c(x) + c(y) - 1 > \left(\frac{\delta}{2} + \frac{\delta+1}{n} + 2\right) - \frac{\delta+1}{n} - 1 = \frac{\delta}{2} + 1 \geq \frac{\delta(\mathcal{T})}{2} + 1.$$

The closed braid K satisfies the conditions in Theorem 1.12 and Corollary 1.13. Thus, $\mathcal{T}$ admits a minimum genus Bennequin surface with exactly $\delta(\mathcal{T})$ negative bands, even though the Bennequin surface F_w may not have the minimum genus $g(\mathcal{T})$.

For suitable choices of x and y, we can easily make $\mathcal{T}$ non-fibered:

Let $x = (\sigma_{1,3}\sigma_{2,4}\sigma_{1,3}\sigma_{2,4})^{N+1}\sigma_{1,3} \in B_4$ for $N \geq \frac{3\delta+9}{4}$ and $y \in B_4$ be a braid word in $\{\sigma_{1,3}^{\pm 1}, \sigma_{2,4}^{\pm 1}\}$ containing δ negative band generators. The closure K of the 4-braid $w = xy$ realizes the braid index $b(\mathcal{T})$ of $\mathcal{T}$. Since the Bennequin surface F_w is not connected, the Alexander polynomial of $\mathcal{T}$ is zero (see [25, Proposition 6.14]). In particular, $\mathcal{T}$ is not fibered. Using [22, Lemma 4.13] we obtain $c(x) \geq N$. The above argument shows that K satisfies the assumptions of Theorem 1.12 and Corollary 1.13.

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